Rhizomes of Ranked Posets

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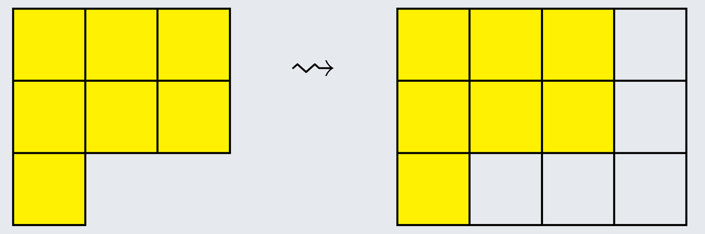
Abstract

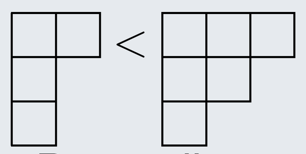
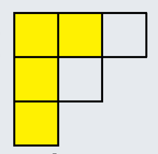
We introduce and study the notion of a *rhizome* for a ranked, partially ordered set (or poset) where each set of fixed rank is finite. A rhizome is defined as a minimal size subset of the elements of rank n such that each of the elements of rank $n+1$ covers at least one element of the rhizome. Given a poset $\mathcal{P}$ with ranked parts $\mathcal{P}_n$, we consider the function $r_{\mathcal{P}} : \mathbb{N} \to \mathbb{N}$ which gives the size of a rhizome for $\mathcal{P}_n$, and study this function for examples like the Boolean lattice and Young's lattice.

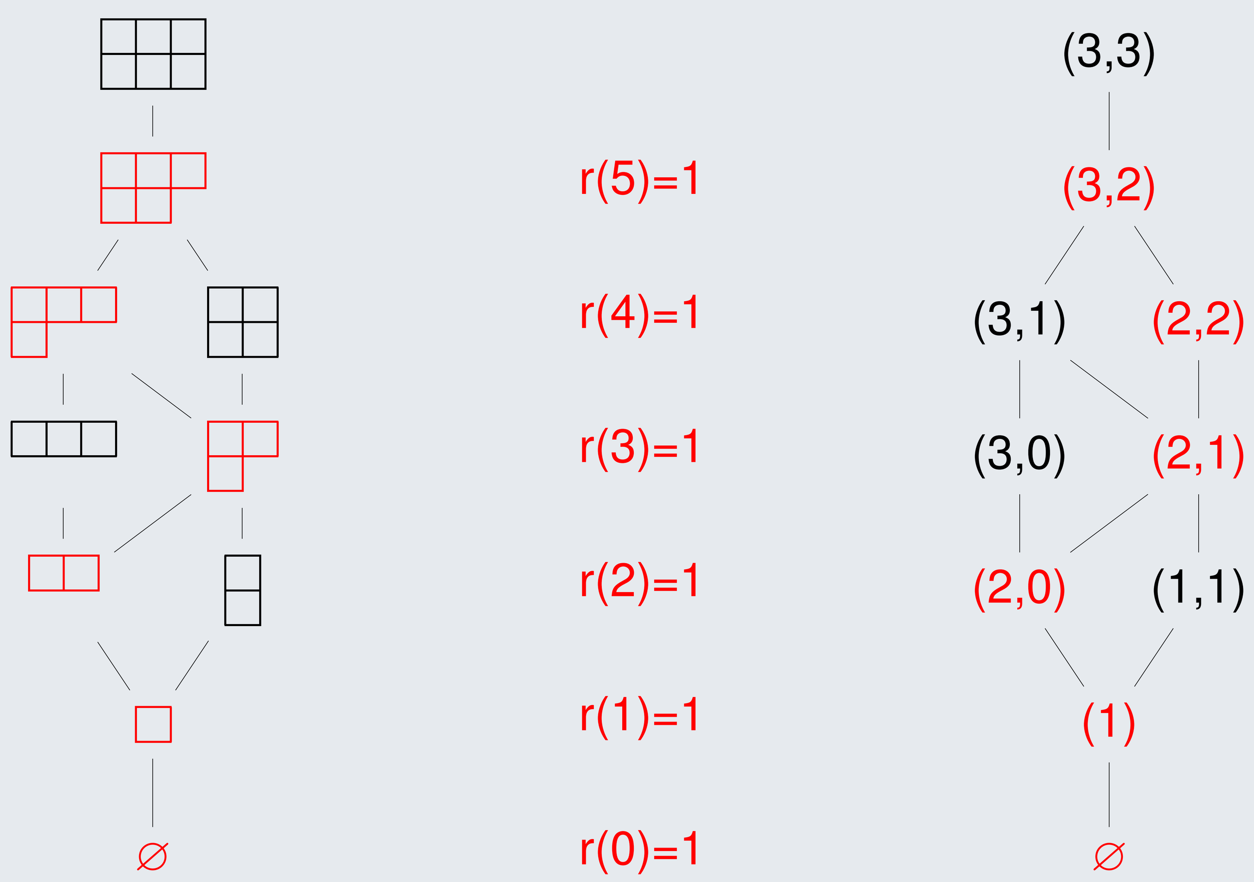
Motivation: Partitions

- **Definition:** A partition λ of a non-negative integer n is a sequence of integers $\lambda = (\lambda_1, \lambda_2, ...)$, $\lambda_i \geq 0$, satisfying $\lambda_1 \geq \lambda_2 \geq ...$ and $\sum_i \lambda_i = n$. If λ is a partition of n, we write $|\lambda| = n$. The "empty" partition consisting of all 0's is denoted $\varnothing$.
- **Example:** Some integer partitions of 7 are: $(4,3)$, $(5,1,1)$, $(3,3,1)$.
- We can represent a partition λ as a Young diagrams, which is a northwest-justified set of boxes where each row i has λ_i boxes.
- Partitions restricted to the dimensions of a $k \times l$ rectangle are important in areas of math such as *number theory, combinatorics, representation theory, and algebraic geometry*. These are λ such that there can be no more than k parts of the partition and each $\lambda_i \leq l$.
- **Example:** The Young Diagram of $(3,3,1)$ fits inside a $k \times l$ rectangle provided $k, l \geq 3$.



Posets of Bounded Partitions

- **Definition:** A poset is a set $\mathcal{P}$ together with a partial order relation $\geq$. This means that for each pair of elements $a, b \in \mathcal{P}$, we have $a \leq b$, $a \geq b$, or a and b may have no relationship at all.
- The set of all partitions that fit into a $k \times l$ rectangle is a poset $P(k,l)$ with ordering of elements λ, μ defined by: $\lambda \leq \mu$ if the Young Diagram of λ "fits" within the Young Diagram of μ, also known as containment. For instance:  since we have the containment 
- **Example:** Poset diagrams for $P(2,3)$ with rhizomes in red:



Rhizomes

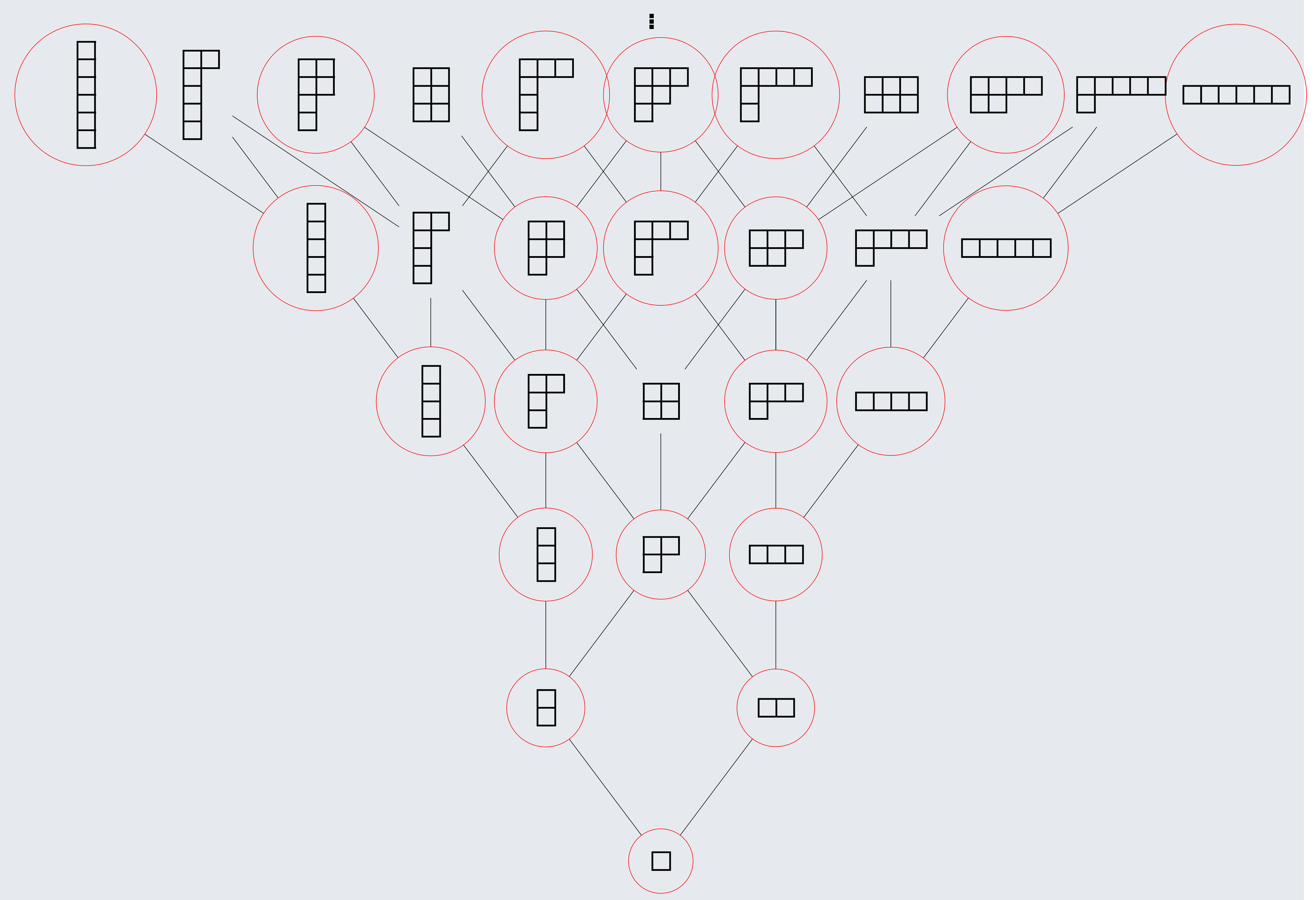
- **Definition:** A level, $\mathcal{P}_n$, of a ranked poset $\mathcal{P}$ consists of the subset of elements of rank n.
- **Example:** If $\mathcal{P}$ is a poset of (bounded) partitions, then $\mathcal{P}_n$ consists of the partitions λ of n, that is those with $|\lambda| = n$.
- **Definition**: If $x \in \mathcal{P}_n$ and $y \in \mathcal{P}_{n+1}$ with $x \leq y$, then we say that λ is a child of x. Equivalently, x is a parent of y.
- Let the function $C : \mathcal{P}_n \to \mathcal{P}_{n+1}$ return the set of children of a given $\lambda \in \mathcal{P}_n$. For $X \subseteq \mathcal{P}_n$, we write

$$C(X) := \bigcup_{x \in X} C(x).$$

- **Definition:** If $X \subseteq \mathcal{P}_n$ is such that $C(X) = \mathcal{P}_{n+1}$, then the set X is called a generating set of $\mathcal{P}_{n+1}$. The smallest number $r(n)$ such that there exists $X = \{x_1, \ldots, x_{r(n)}\} \subseteq \mathcal{P}_i$ with $C(X) = \mathcal{P}_{i+1}$ is called the is rhizal number associated to $\mathcal{P}_n$. A generating set of size $r(n)$ is called a rhizome (see $P(2,3)$ diagram at left for examples).

Young's Lattice

- **Definition:** Young's lattice, $\mathbb{Y}$, is a ranked poset on the set of all non-negative integer partitions ordered by containment. It is countably infinite and has levels $\mathbb{Y}_n$ which consist of all partitions of n. Thus $|\mathbb{Y}_n| = p(n)$, where $p(n)$ is the number of integer partitions of n.
- **Example:** In the following part of Young's lattice, a rhizomes on each level is circled. These sets are not always unique.



Easy Bounds for $r_{\mathbb{Y}}$

- **Theorem (Hardy–Ramanujan):** The function $p(n)$ is asymptotically equivalent to $\frac{1}{4n\sqrt{3}} e^{\pi\sqrt{2\frac{n}{3}}}$.
- **Proposition:** The loosest bounds we begin with are

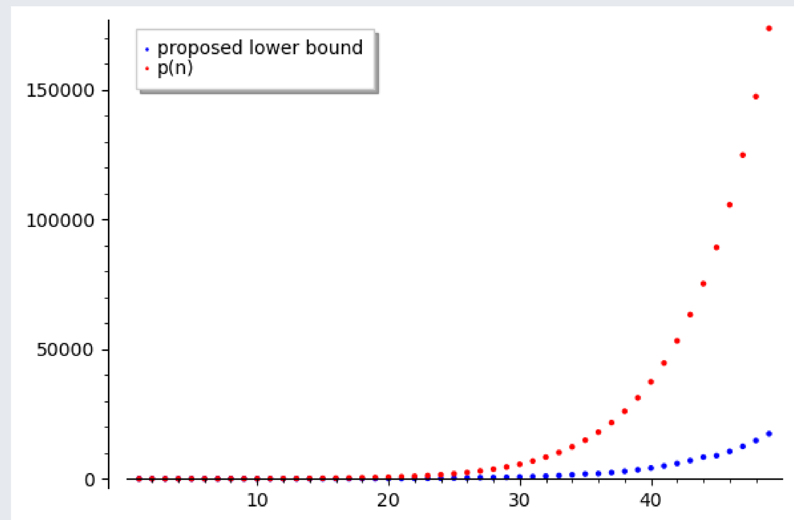
$$1 \leq r_{\mathbb{Y}}(n) \leq p(n)$$

.

Main result

- The function $t(n)$ provides the index j of the largest triangular number T_j which is less than or equal to n.
- **Fact:** $t(n) = \lfloor \frac{-1 \pm \sqrt{1+8n}}{2} \rfloor$.
- **Proposition:** We have a lower bound of

$$\lceil \frac{p(n)}{t(n)+1} \rceil \leq r_{\mathbb{Y}}(n)$$



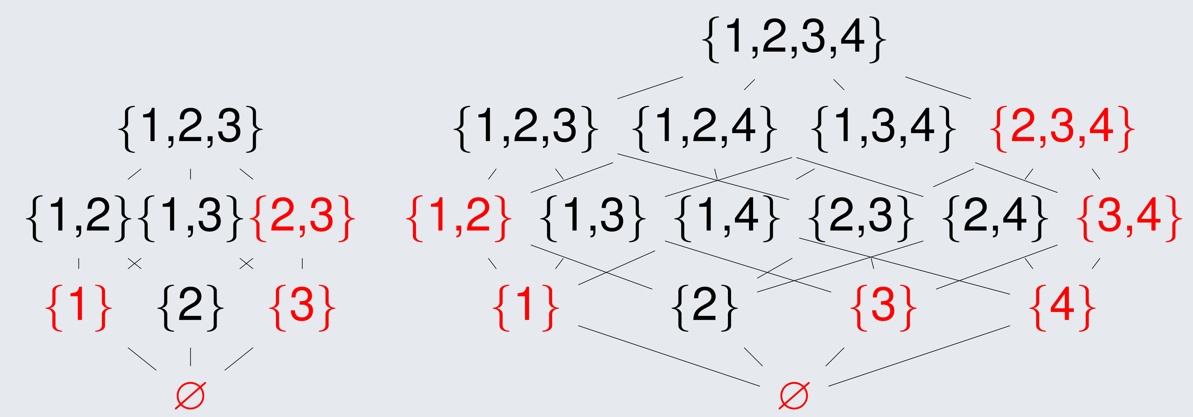
Rhizal Sequences

- **Definition:** We define a rhizal sequence $\mathcal{C}(\mathcal{P})$ for a poset $\mathcal{P}$ as the sequence of rhizal numbers $r(1), r(2), ..., r(n-1), \ldots$, i.e. the size of the rhizomes on each level. Note that if $\mathcal{P}$ is finite, then the highest level n has no rhizomes.
- **Proposition:** $\mathcal{C}(\mathbb{Y}) =$

$$1, 1, 2, 3, 5, 7, 11, 15, 22, 30, 42, ...$$

Work in Progress: Boolean Lattices

- **Definition:** $\mathcal{B}^n$ is the poset consisting of all subsets of $\{1, 2, ..., n\}$, partially ordered by inclusion.
- **Proposition:** Let $\mathcal{C}(B) = (\mathcal{C}(\mathcal{B}^0), \mathcal{C}(\mathcal{B}^1), \mathcal{C}(\mathcal{B}^2), ...)$. We find that $\mathcal{C}(B) =$ $0, 1, 1, 1, 1, 2, 1, 1, 3, 2, 1, 1, 4, 4, 3, 1, 1, 5, 7, 7, 3, 1, 1, 6, 9, 14, 7, 4, 1, ...$
- **Examples:** $\mathcal{C}(\mathcal{B}^3) = 1, 2, 1$ and $\mathcal{C}(\mathcal{B}^4) = 1, 3, 2, 1$



Resources

- Melczer, Stephen, Greta Panova, and Robin Pemantle. "Counting partitions inside a rectangle." *SIAM Journal on Discrete Mathematics* 34, no. 4 (2020): 2388-2410.
- Stanley, Richard P. "Enumerative Combinatorics Volume 1 (2nd edition)." Cambridge studies in advanced mathematics (2011).
- Zeilberger, Doron. "Kathy O'Hara's constructive proof of the unimodality of the Gaussian polynomials." *The American Mathematical Monthly* 96, no. 7 (1989): 590-602.