Classification of Sub-Universal Quantum Gatesets

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Outline

Question

When do sub-universal quantum gatesets admit efficient classical simulation? How can we classify them?

Gottesman-Knill

Theorem 1 (Gottesman-Knill '98)

A quantum circuit comprised of the following:

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So what else is easy, and how can we find it?

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- What is $SU(2)$?
  - Special unitary group: $SU(2) \subset U(2) \subset GL_2(\mathbb{C})$.
  - (Note, we usually consider $SU(2)\ /\mathbb{Z}_2$, which happens to be isomorphic to $SO(3)$ via the so-called "double-cover isomorphism".)
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- What is a dense subgroup?
  - If H is a subgroup of G, then $\overline{H} = G$ implies H is dense in G.
  - ie, for any sequence of matrices in H, its limit is in G, and there are no points in G that are not either in H or a limit point of H.
  - $\overline{(\mathbb{Q}, +)} = (\mathbb{R}, +)$

Bloch Sphere

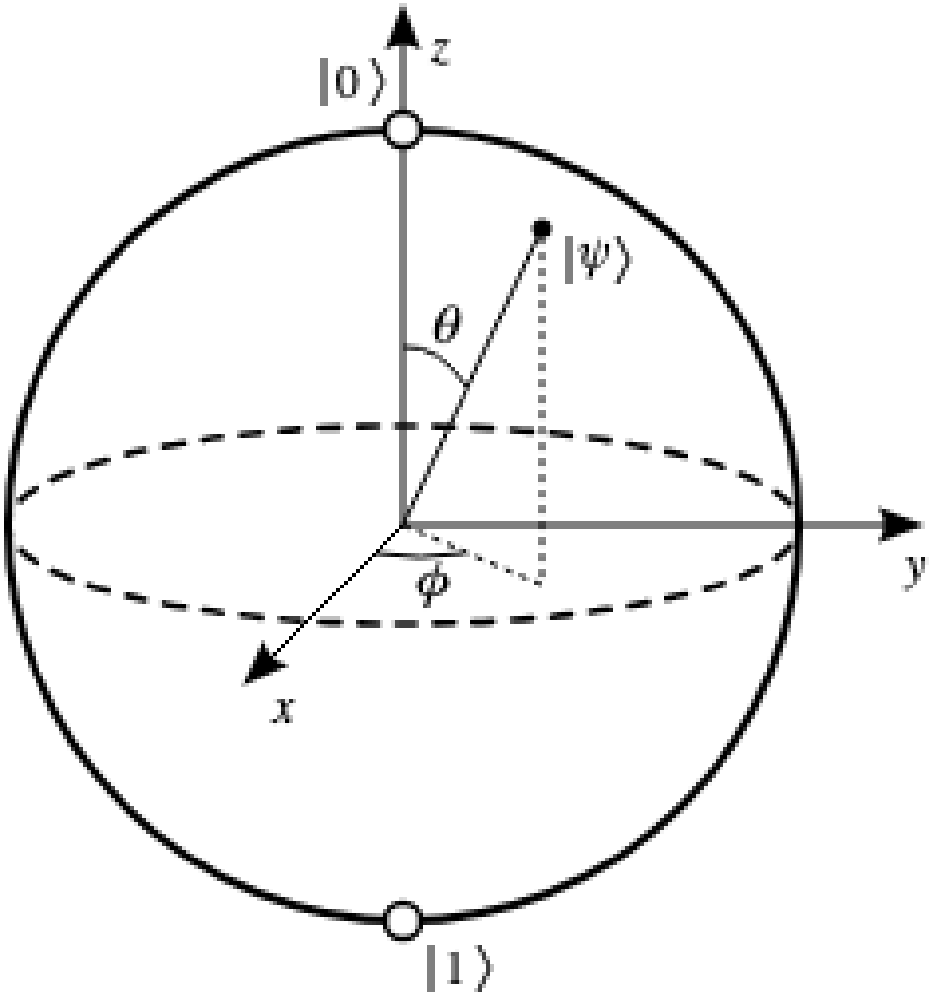
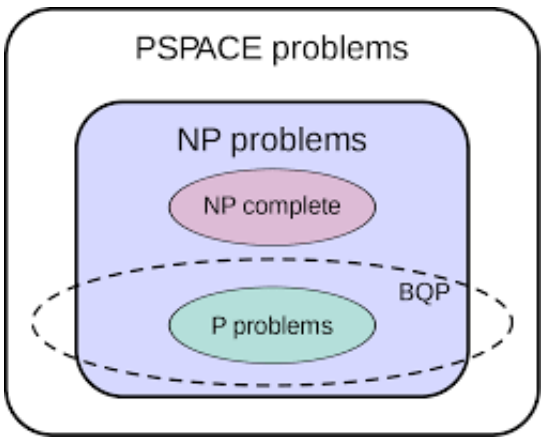


Figure 1: Bloch Sphere

BQP and P

- Rachel believes and Alexey would like it to be known he does not see evidence either way regarding $BQP \nsubseteq P$, so generic quantum circuits likely cannot be easily simulated on classical computers.



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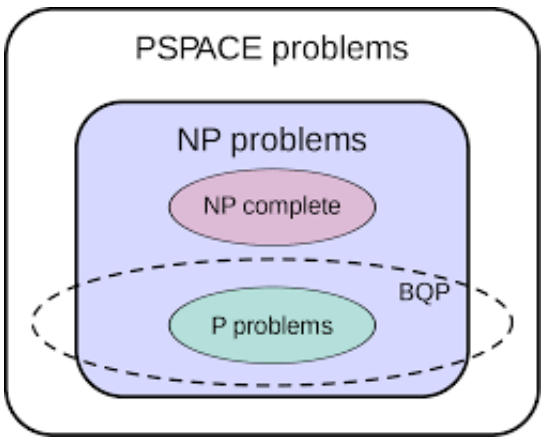

- Any subgroup that is not dense is not universal, and might be easier to simulate classically.
- How might we classify these sub-universal gatesets?

Boolean Classification

As it turns out, researchers have been trying to classify sub-universal computation for a long time...

Figure 2: Emil Post circa 1940s

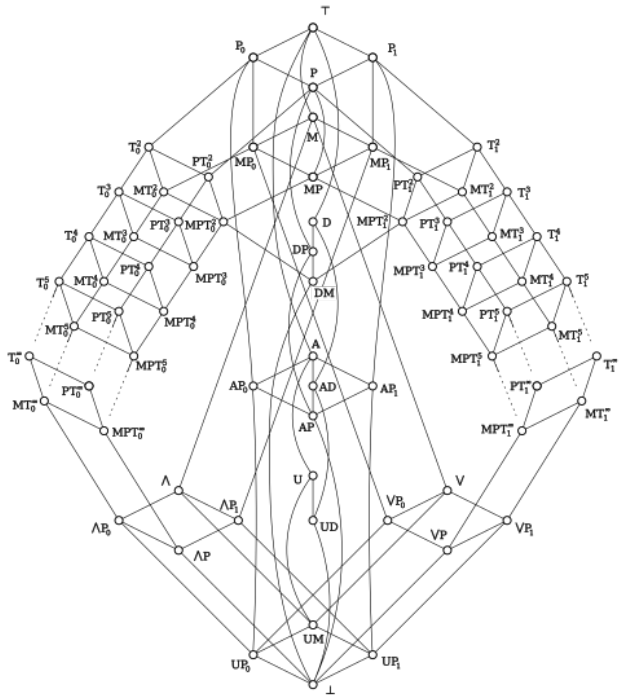

Figure 3: ...and his lattice ('41)

Post's Lattice Continued

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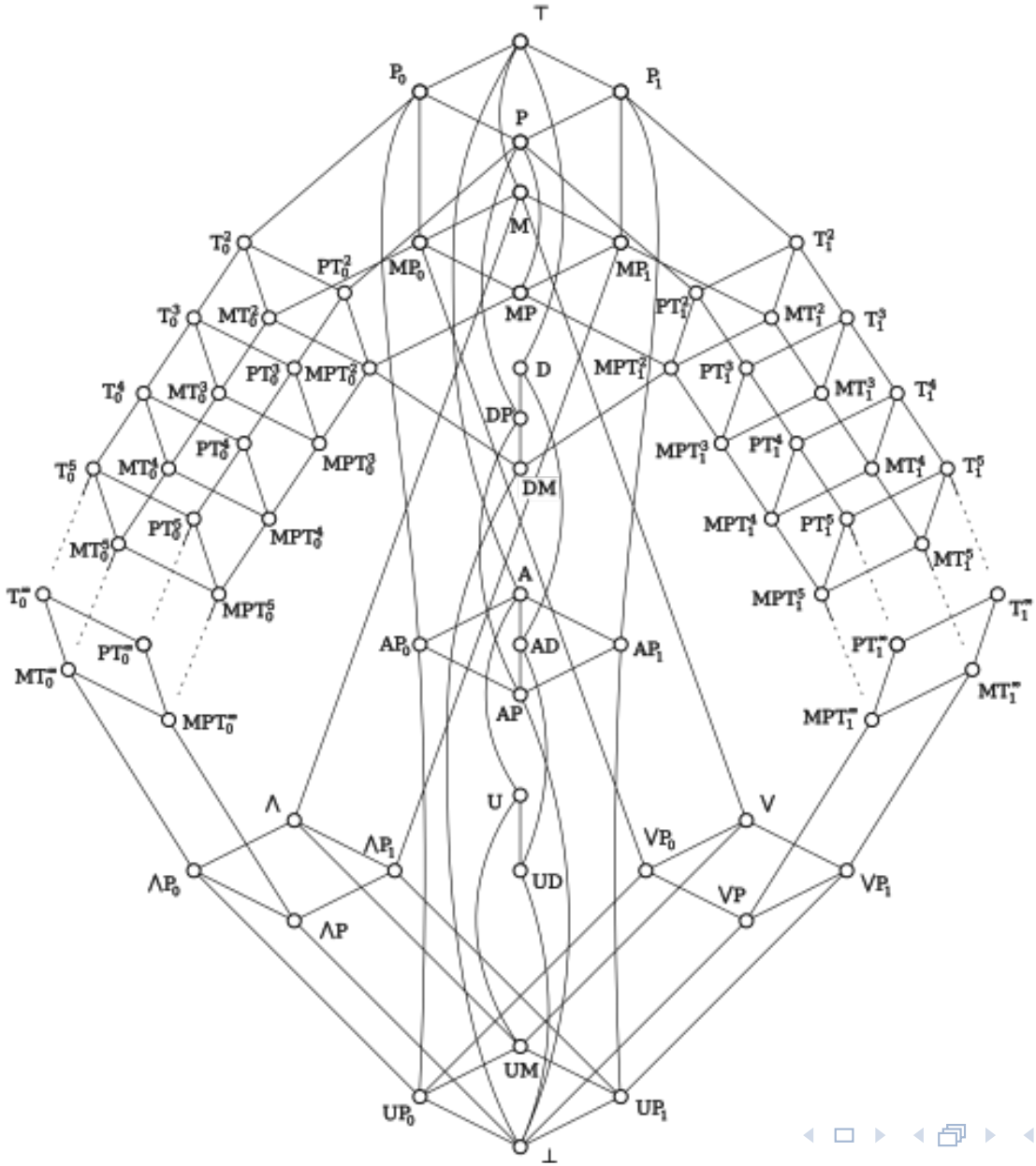
- The 5 *maximal (proper) clones* famously are:
  - Affine functions (over $\mathbb{F}_2$). Interestingly, this class exists in the generalization of Post's lattice to arbitrary finite domains if and only if the domain size is a prime power, due to the existence of a field of that size
  - 0-preserving ($f(0, 0, \ldots, 0) = 0$) and 1-preserving functions
  - Self-dual ($f(x_1, \ldots, x_n) = \neg f(\neg x_1, \ldots, \neg x_n)$)
  - Monotone functions $x \leq y \implies f(x) \leq f(y)$
- Surprisingly, there exists a complete classification of all clones (not just the maximal ones) of Boolean functions: two countable families and a few dozen exceptions:

An easier picture



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- Since the 90s, researchers have been trying to extend this result to quantum computing.
- Aaronson, Grier, and Schaeffer [Aaronson et al., 2015] classified all classical reversible bit operations.
- Grier and Schaeffer [Grier and Schaeffer, 2022] continued their work to partition all Clifford gates on n qubits into 57 classes.



Classification of Clifford Gates over Qubits

Definition 3 (Clifford Gateset)

The Clifford group on n qubits normalizes the Pauli group of n-length Pauli strings in $SU(2^n)$.
That is, $C \in \mathcal{C}$ iff $\exists P \in \mathcal{P}^n$ such that:

$$CPC^\dagger \in \mathcal{P}$$

Where $\mathcal{P}^n = \pm\{I, X, Y, Z\}^{\otimes n}$
Generators of this group include $\{S, H, CNOT\}$, $\{R_X, R_Z\}$, $\{H, CZ, X, Z\}$

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There are 57 total classes, 30 "degenerate". The remaining 27 are more interesting.

Closure

Definition 4 (Closure)

A class $\mathcal{C}$ of Clifford gates is **closed** under the following (for $f, g \in \mathcal{C}$)

1. Composition: $f \circ g \in \mathcal{C}$
2. Tensor Product: $f \otimes g \in \mathcal{C}$
3. SWAP
4. Arbitrary ancilla (no magic states): if $f \in \mathcal{C}, \exists h \in \mathcal{C}, \exists \left|\psi\right\rangle$ such that:

$$f(\left|x\right\rangle \otimes \left|\psi\right\rangle) = g(\left|x\right\rangle) \otimes \left|\psi\right\rangle, \forall x$$

It follows that $\mathcal{C}$ is closed under inverse and that $I \in \mathcal{C}$.

Lattice

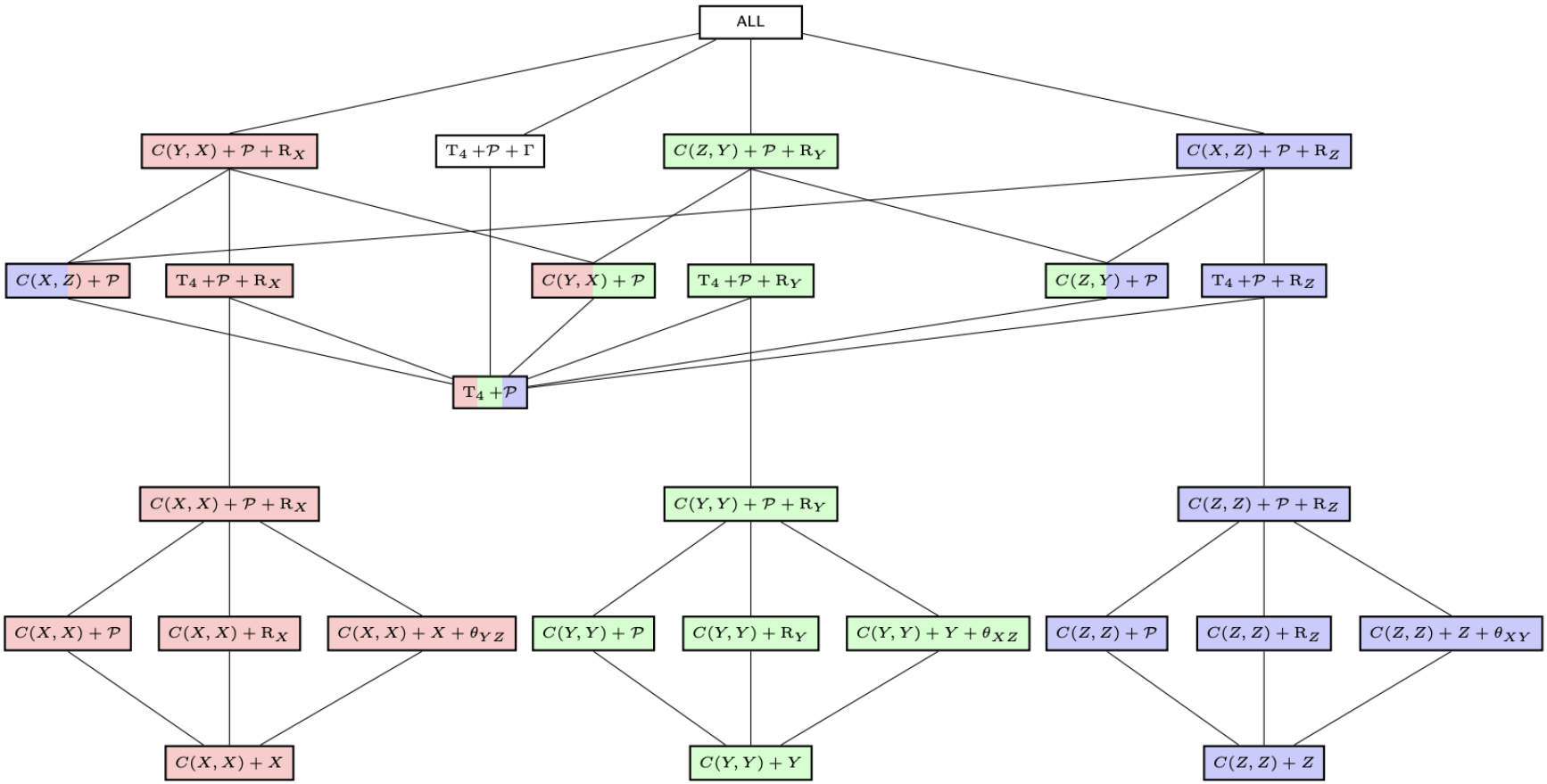

Figure 2: The inclusion lattice of non-degenerate Clifford gate classes. Red, green, blue denote X-, Y-, and Z-preserving, respectively.

More Classification

So how do we actually distinguish these classes?

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So how do we actually distinguish these classes? Invariants!

- X, Y, Z Preserving - mapping basis states to themselves
  - ex: $X, CNOT, T_4$ are Z-preserving.
- "Degenerate" - each input effects exactly one output
  - All degenerates are generated by single-qubit gates and $SWAP$.
- X, Y, Z "Degenerate" - flipping any bit in the related basis input flips exactly one bit of the output.
  - ex: R_Z, $\mathcal{P}$
- X, Y, Z "Orthogonal" - built from Z-preserving gates and T_4
- "Egalitarian" - symmetric across bases
  - ex: Γ, T_4

Our Work So Far

- We are trying to understand how non-dense subgroups of $SU(2)/\mathbb{Z}_2 \cong SO(3) \cong PU(2)$ and $SU(4)/\text{Phase} \cong PU(4)$ are classified.

SO(3)

- For $SO(3)$, there's a technique based on the counting of pairs (g, p) s.t. $g \in G \backslash \{1\}$ fixes p. It turns out that $SO(3)$ has a complete classification of (maximal) finite subgroups:
  - Dihedral groups D_n of arbitrary order – rotate around an axis and allow flipping,
  - S_4 – because it's the symmetry group of a cube/octahedron,
  - even A_5 – the symmetry group of an isocahedron/dodecahedron.

Figure 4: Octahedron and cube are duals

SU(2)

- In representation theory, there's a known surjective homomorphism $\Phi\colon SU(2) \to SO(3)$, s.t. $SU(2)/SO(3) \cong \mathbb{Z}_2$. It's called the double cover homomorphism, because $\forall y \in SO(3)\ |\Phi^{-1}(y)| = 2$
- It's very useful for studying SU(2), because you can employ a lot of geometric insights to study SO(3), and then lift your results through this homomorphism
- Note: The homomorphism Φ is actually the action of $SU(2)$ on $\mathrm{Span}_{\mathbb{R}}(i\sigma_x, i\sigma_y, i\sigma_z)$ by conjugations
- Using this techniques, one can classify maximal subgroups of $SU(2)$:
  - binary dihedral groups $\mathcal{BD}_n$ of arbitrary order
  - the binary cube/octahedral group $\mathcal{BO}$
  - the binary dodecahedron/icosahedron group $\mathcal{BI}$

SU(¿2), SO(¿3)

- The double cover homomorphism no longer exists to aide us to understand one group through another
- In 4 dimensions, there are 6 regular polyhedra, including one with 1200 faces!
- Though to be fair, it's also known that for 5d+, there are only two (the symplex and the cube)
- Compared to the 2-dimensional $PU(2)$, $PU(4)$ is 14-dimensional, making the involved geometric structure slightly more involved

Conclusion

We are investigating what connection there is, if any, between group structure of sub-universal quantum gates and easy classical simulation.

- We saw the Gottesman-Knill and Valiant results
- We discussed how gatesets have been classified in classical computing
- We followed the attempts to extend these results into quantum computing
- We saw the areas of future work to continue classifying quantum gatesets

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